ADAPTATION OF INITIAL CONDITIONS FOR INTERNAL WAVES IN A SLIGHTLY COMPRESSIBLE FLUID

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1. Let us consider the linearized system of hydrodynamics equations for a stratified liquid

$$\begin{gathered}\rho_0 u_t + p_x = 0,\ p_t - \rho_0 g w + \rho_0 c^2(u_x + v_y + w_z) = 0,\\ \rho_0 v_t + p_y = 0,\ \rho_0 w_t + p_z + g\rho = 0,\\ \rho_t + w d\rho_0/dz + \rho_0(u_x + v_y + w_z) = 0.\end{gathered} \tag{1.1}$$

Here u, v, w are velocity components, p is the pressure, $\rho_0 = \rho_0(z)$ is the unperturbed density, ρ is the density perturbation, g is the acceleration of gravity, and c is the speed of sound.

By a formal passage to the limit $c \to \infty$, the system (1.1) is converted into a system of internal wave equations

$$\begin{gathered}\rho_0 u_t + p_x = 0,\ \rho_0 v_t + p_y = 0,\\ \rho_0 w_t + p_z + g\rho = 0,\ u_x + v_y + w_z = 0,\ \rho_t + w d\rho_0/dz = 0.\end{gathered} \tag{1.2}$$

It is impossible to give the initial values $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{p}$, $\tilde{\rho}$ arbitrarily for (1.2). Besides the incompressibility condition

$$\tilde{Q} = \tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0 \tag{1.3a}$$

they should satisfy the equation

$$\Delta\tilde{p} + g\tilde{\rho}_z - \rho_0^{-1} d\rho_0/dz\,(\tilde{p}_z + g\tilde{\rho}) = 0. \tag{1.3b}$$

Consequently, the Cauchy problem for (1.2) requires "consistent" initial data that satisfy the conditions (1.3). At the same time the initial data for the original system (1.1) can be given arbitrarily. Then for c >> 1, when the solution of the system (1.1) is close to the solution of the limit system (1.2) while the initial data u^c, v^c, w^c, p^c, ρ^c are arbitrary and do not satisfy conditions (1.3), a process of "adaptation" of the initial data to these conditions should occur. It is natural to propose that this transition process include sound wave radiation, occur at the times $\tau = L/c$ (L is the characteristic dimension of the problem), and result in consistent initial conditions $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{p}$, $\tilde{\rho}$ for which the velocity field is made vortical while the pressure and density turn out to be related by relationship (1.3b).

The question of how to determine the consistent data $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{p}$, $\tilde{\rho}$ from the original initial functions also occurs naturally. Let us note the analogy between such a formulation of the problem and the classical paper [1]. The steady motion in the problem of a geostrophic wind is also purely vortical, where the velocity components and the pressure are connected by the relationships $u = -(2\omega_z\rho)^{-1}\partial p/\partial y$, $v = -(2\omega_z\rho)^{-1}\partial p/\partial x$, where $\omega_z = \omega_0 \sin\theta$, and θ is the latitude. If the hydrodynamic field varies in some domain, its "adaptation" occurs accompanied by wave radiation in conformity with the equation $\partial^2\varphi/\partial t^2 = c^2\Delta\varphi - 4\omega_z^2\varphi$ (φ is the velocity potential and c is the speed of sound). During this radiation, the potential part of the field "runs away" while the stationary value of the stream function ψ and its corresponding pressure $\bar{p}$ are determined uniquely from the initial data.

2. Let us give the solution of the formulated problem for the simplest case of a limitless exponential medium, $\rho_0(z) = \rho_* \exp(-\varkappa z)$. Then (1.1) and (1.2) reduce to systems with constant coefficients and their solutions are written in quadratures. We assume here that the initial data for (1.1) are localized in a certain domain of diameter L; then the

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characteristic time of sound wave propagation is $\tau_c = L/c$. On the other hand, the time scale for (1.2) is the period of the buoyancy $T = 2\pi/N$ ($N = [-g\rho_0^{-1}\partial\rho_0/\partial z]^{1/2} = \sqrt{\varkappa g}$ is the Brent-Waisayla frequency). We furthermore assume that $\tau_c << T$, i.e., $L(\varkappa g)^{1/2}$.

Let us formulate the main result of the research. We expand the fluid flow field $\rho_0(z)\mathbf{v}(\mathbf{x}, t)$ into potential and vortical parts

$$\begin{gathered}\rho_0\mathbf{v}(\mathbf{x}, t) = \rho_0\mathbf{v}_p(\mathbf{x}, t) + \rho_0\mathbf{v}_v(\mathbf{x}, t),\\ \rho_0\mathbf{v}_p(\mathbf{x}, t) = \nabla\varphi(\mathbf{x}, t),\ \mathbf{v}_v(\mathbf{x}, t) = \operatorname{rot}\boldsymbol{\psi}(\mathbf{x}, t).\end{gathered} \tag{2.1}$$

This decomposition differs from the ordinary one: it is necessary that the velocity field $\mathbf{v}_v$ be vortical and the fluid flow field $\rho_0\mathbf{v}_v$ be potential. To construct the decomposition (2.1), we write $\mathbf{v}_v = \mathbf{v} - \rho_0^{-1}\nabla\varphi$, and from the condition $\operatorname{div}\mathbf{v}_v = 0$, we obtain for φ the equation $\Delta\varphi - (\nabla\rho_0, \nabla\varphi) = \rho_0 \operatorname{div}\mathbf{v}$.

The condition of subsidence at infinity assures uniqueness of the solution φ. We analogously set for the initial values of $\mathbf{v}^0(\mathbf{x})$

$$\rho_0\mathbf{v}^0(\mathbf{x}) = \rho_0\mathbf{v}_p^0(\mathbf{x}) + \rho_0\mathbf{v}_v^0(\mathbf{x}),\ \rho_0\mathbf{v}_p^0(\mathbf{x}) = \nabla\varphi^0(\mathbf{x}),\ \mathbf{v}_v^0(\mathbf{x}) = \operatorname{rot}\boldsymbol{\psi}^0(\mathbf{x}). \tag{2.1a}$$

Let $\tilde{p}(\mathbf{x})$ be defined from the initial value $\rho^0(\mathbf{x})$ according to (1.3b) and the additional condition $\tilde{p} \to 0$ as $|\mathbf{x}| \to \infty$. The main result of the research is a theorem by virtue of which the velocity field becomes purely vortical during adaptation of the initial conditions while the pressure is adjusted by the density.

<u>Theorem A</u>. For fixed t, $\mathbf{x}$ and $c \to \infty$ the solution of the system (1.1) tends to the solution of (1.2) satisfying the "consistent" initial conditions

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_v^0(\mathbf{x}) = \operatorname{rot}\boldsymbol{\psi}^0(\mathbf{x}),\ \rho(\mathbf{x}, 0) = \rho^0(\mathbf{x}),\ p(\mathbf{x}, 0) = \tilde{p}(\mathbf{x}). \tag{2.2}$$

According to Theorem A, for arbitrarily small $t = t_0$, large values of c can be indicated such that the field of $\mathbf{v}, p, \rho$ at the time t_0 would be arbitrarily close to (2.2). To describe the adaptation process more exactly we introduce the "fast" time $\tau = ct$.

<u>Theorem B.</u> For fixed τ, $\mathbf{x}$ and $c \to \infty$, the functions $\mathbf{v}_v(\mathbf{x}, \tau)$ and $\rho(\mathbf{x}, \tau)$ tend to the initial functions $\mathbf{v}_v^0(\mathbf{x})$ and $\rho^0(\mathbf{x})$, the function $\mathbf{v}_p(\mathbf{x}, \tau)$ tends to the limit value $\mathbf{v}_1(\mathbf{x}, \tau)$ and $p(\mathbf{x}, \tau) = cp_1(\mathbf{x}, \tau) + p_0(\mathbf{x}, \tau)$. For fixed $\mathbf{x}$ and $\tau \to \infty$, the functions $\mathbf{v}_1(\mathbf{x}, \tau)$ and $p_1(\mathbf{x}, \tau)$ tend to zero, while the function $p_0(\mathbf{x}, \tau)$ tends to the limit $\tilde{p}(\mathbf{x})$, where $\tilde{p}(\mathbf{x})$ is determined by means of $\rho^0(\mathbf{x})$ by virtue of (1.3b).

Let us turn attention to the following circumstance. As is known, the system (1.2) reduces to one equation, say, for the vertical velcoity w:

$$\partial^2/\partial t^2[\Delta w - g^{-1}N^2\partial w/\partial z] + N^2\Delta_h w = 0 \tag{2.3}$$

($\Delta_h = \partial^2/\partial x^2 + \partial^2/\partial y^2$) requiring assignment of two arbitrary initial fucntions. At the same time, (1.2) requires the assignment of five initial functions subject to the two conditions $\operatorname{div}\mathbf{v}^0(\mathbf{x}) = 0$ and (1.3b), i.e., permits arbitrary assignment of three initial functions. Where is the excess initial function lost when going from (1.2) to (2.3)? The fact is that (1.1) and (1.2) have still another solution, stationary horizontal vortices, in addition to the acoustic and internal waves for (1.1) and the internal waves for (1.2). Namely, if the condition $\partial u^0/\partial x + \partial v^0/\partial y = 0$ is satisfied, then as is easily seen, the functions $u(\mathbf{x}, t) = u^0(\mathbf{x})$, $v(\mathbf{x}, t) = v^0(\mathbf{x})$, $w(\mathbf{x}, t) \equiv p(\mathbf{x}, t) \equiv \rho(\mathbf{x}, t) \equiv 0$ are exact solutions of the systems (1.1) and (1.2). To filter these vortices out we can decompose the initial field of horizontal velocities into potential and vortical parts by setting $u^0(\mathbf{x}) = u_p^0(\mathbf{x}) + u_v^0(\mathbf{x})$, $v^0(\mathbf{x}) = v_p^0(\mathbf{x}) + v_v^0(\mathbf{x})$($\partial u_p^0/\partial y = \partial v_p^0/\partial x$, $\partial u_v^0/\partial x = -\partial v_v^0/\partial y$). Then the vortical part of the horizontal velocity field is conserved while the remaining potential field satisfies the condition $\partial u^0/\partial y = \partial v^0/\partial x$ that can be considered as the third consistency condition for reducing the Cauchy problem for (1.2) from five equations to the second order equation (2.3) in t.

3. Let us introduce the function $Q(\mathbf{x}, t) = \operatorname{div}\mathbf{v} = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z$. Then (1.1) will reduce to the system

$$\begin{gathered}\rho_0 Q_t + \rho_0' w_t + \Delta p + g\rho_z = 0,\quad \rho_t + \rho_0' w + \rho_0 Q = 0,\\ \rho_0 w_t + p_z + g\rho = 0,\ p_t - \rho_0 g w + c^2\rho_0 Q = 0,\end{gathered} \tag{3.1}$$

which can be solved to yield $u(\mathbf{x}, t)$ and $v(\mathbf{x}, t)$ when utilizing conservation of the horizontal vorticity $\omega(\mathbf{x}, t)$: $\omega(\mathbf{x}, t) = \partial u(\mathbf{x}, t)/\partial y - \partial v(\mathbf{x}, t)/\partial x \equiv \partial u^0/\partial y - \partial v^0/\partial x = \omega^0(\mathbf{x})$, i.e., from the equations $\partial u(\mathbf{x}, t)/\partial x + \partial v(\mathbf{x}, t)/\partial y = Q(\mathbf{x}, t) - \partial w(\mathbf{x}, t)/\partial z$, $\partial u(\mathbf{x}, t)/\partial y - \partial v(\mathbf{x}, t)/\partial x = \omega^0(\mathbf{x})$.

Let us reformulate Theorems A and B in terms of the system (3.1). To do this, we find the potential and vortical parts of the function $w(\mathbf{x}, t)$. From (2.1) there follows $\operatorname{div}(\rho_0\mathbf{v}) = \rho_0 Q + \rho_0' w = \Delta\varphi$, i.e., the potential φ is determined from the Poisson equation $\Delta\varphi = \rho_0 Q +$ by means of the known functions Q, w, from which $\varphi = \Delta^{-1}(\rho_0 Q + \rho_0' w)$ (Δ^{-1} is an integral operator inverse to the Laplace operator). Evidently,

$$\rho_0 w_{\mathrm{p}} = \partial\varphi/\partial z = \Delta^{-1}\left[(\rho_0 Q)_z' + (\rho_0' w)_z'\right],\ w_{\mathrm{v}} = w - w_{\mathrm{p}}. \tag{3.2}$$

Consequently, Theorems A and B are formulated as follows:

Theorem A'. For fixed t, $\mathbf{x}$ and $c \to \infty$ the solution of the system (3.1) tends to the solution

$$\rho_0 w_t + p_z + g\rho = 0,\ \Delta p + \rho_0' w + g\rho_z = 0,\ \rho_t + w\rho_0' = 0, \tag{3.1'}$$

satisfying the additional condition Q = 0 and the initial conditions

$$\begin{gathered} w(\mathbf{x}, 0) = w^0(\mathbf{x}) - \rho_0^{-1}\Delta^{-1}(\rho_0 Q^0 + \rho_0' w^0)_z', \\ \rho(\mathbf{x}, 0) = \rho^0(\mathbf{x}),\ p(\mathbf{x}, 0) = \widetilde{p}(\mathbf{x}) \end{gathered} \tag{3.3}$$

($\widetilde{p}(\mathbf{x})$. is determined as the solution of (1.3b)).

Theorem B'. For fixed $ct = \tau$, $\mathbf{x}$ and $c \to \infty$ the functions $w_{\mathrm{v}}(\mathbf{x}, \tau)$ and $\rho(\mathbf{x}, \tau)$ tend to the initial values $w_{\mathrm{v}}^0(\mathbf{x})$ and $\rho^0(\mathbf{x})$, the functions $Q(\mathbf{x}, \tau)$ and $w_{\mathrm{p}}(\mathbf{x}, \tau)$ to the limit values $Q_1(\mathbf{x}, \tau)$ and $w_{\mathrm{p}1}(\mathbf{x}, \tau)$ while $p(\mathbf{x}, \tau) = cp_1(\mathbf{x}, \tau) + p_0(\mathbf{x}, \tau)$. For fixed $\mathbf{x}$ and $\tau \to \infty$ the functions $Q_1(\mathbf{x}, \tau), w_{\mathrm{p}1}(\mathbf{x}, \tau), p_1(\mathbf{x}, \tau,$ tend to zero while $p_0(\mathbf{x}, \tau)$ tends to the function $\widetilde{p}(\mathbf{x})$, the functions solution of (1.3b).

To prove Theorems A' and B' we go over from (3.1) to a system with constant coefficients by setting $p = \exp(-\varkappa z/2)\rho_*\bar{p}$, $\rho = \exp(-\varkappa z/2)\rho_*\bar{\rho}$, $(Q, w) = \exp(\varkappa z/2)(\bar{Q}, \bar{w})$. Then we obtain for the variables $\bar{Q}$, $\bar{w}$, $\bar{p}$, $\bar{\rho}$ (the bar is henceforth omitted)

$$\begin{gathered} Q_t + \Delta p - \varkappa^2 p/4 + g\rho_z + \varkappa g\rho/2 = 0,\ \rho_t - \varkappa w + Q = 0, \\ w_t + p_z - \varkappa p/2 + g\rho = 0,\ p_t - gw + c^2 Q = 0, \\ (Q, w, p, \rho)|_{t=0} = (Q^0, w^0, p^0, \rho^0). \end{gathered} \tag{3.4}$$

The solution of the system (3.4) is found easily in quadratures. Let us denote the unknown functions Q, w, p, ρ in terms of u_i (i = 1, 2, 3, 4); then we write (3.4) in the form

$$Au = 0,\ u|_{t=0} = u^0, \tag{3.4'}$$

where the elements of the matrix A are differential operators with constant coefficients. Let $\mathscr{E}$ denote the fundamental solution of the equation $(\det A)\varphi = 0$, i.e., the solution of the equation $(\det A)\mathscr{E} = \delta(t)\delta(\mathbf{x})$ that vanishes for t < 0. Then the solution of the system (3.4) has the form

$$u_i(\mathbf{x}, t) = \sum_{j=1}^{4} B_{ij}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left[\mathscr{E}(t, \mathbf{x}) * u_j^0(\mathbf{x})\right]. \tag{3.5}$$

Here * is the convolution operation in the space variables, B_{ij} are cofactors of the elements A_{ij} of the matrix A. The fundamental solution for the system (3.4) is written down in the Appendix (this problem is also of independent interest). By using (3.5) , the solution of the problem (3.4) can be written down and satisfaction of Theorems A' and B' can be verified by a direct calculation. However, this path results in awkward calculations; consequently, we note a simpler proof of Theorems A' and B'.

Let us first examine Theorem B'. In the standard notation ($\partial/\partial t$, $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$) $\to$ (ω, α, β, γ), $k^2 = \alpha^2 + \beta^2 + \gamma^2$, the determinant (3.4) is

$$\det A = \omega^4 - c^2[\omega^2 k^2 + N^2(k^2 - \gamma^2) - \omega^2\varkappa^2/4],\ N^2 = \varkappa g - g^2/c^2. \tag{3.6}$$

Writing down the corresponding cofactors of (3.4): $B_{11} = \omega^3 + \varkappa\omega g + \omega g(\gamma - \varkappa/2)$, $B_{21} = -\varkappa\omega g(\gamma + \varkappa/2) - \omega g(k^2 - \varkappa^2/4)$, $B_{31} = -\varkappa g(k^2 - \gamma^2) - \omega^2(k^2 - \varkappa^2/4)$, $B_{41} = g^2(k^2 - \gamma^2) - g\omega^2(\gamma + \varkappa/2)$; and the expression for $Q(\mathbf{x}, \tau)$ by the Fourier method in terms of the Fourier transforms $Q^0(\mathbf{k}), w^0(\mathbf{k})$,

$p^0(\mathbf{k}), \rho^0(\mathbf{k})$ of the initial data, and also taking into account that the substitution $\xi = \omega/c$ corresponds to the substitution $\tau = ct$, we find

$$\lim_{c\to\infty} Q(\mathrm{x}, \tau/c) = Q_1(\mathrm{x}, \tau) = (2\pi)^{-3} \int_{R^3} \exp\{-i(\mathbf{k}, \mathrm{x})\} \times Q^0(\mathbf{k}) \cos((k^2 + \varkappa^2/4)^{1/2}\tau)\, d\mathbf{k},$$

$$\lim_{\tau\to\infty} Q_1(\mathrm{x}, \tau) = 0 \ (\mathbf{k} = (\alpha, \beta, \gamma) \text{ and } Q^0(\mathbf{k}) = \int_{R^3} \exp\{i(\mathbf{k}, \mathrm{x})\} Q^0(\mathrm{x})\, d\mathrm{x}).$$

Analogously, using the formulas for the cofactors B_{12}, B_{22}, B_{32}, B_{42}, and passing to the limit in the Fourier representation for $w(\mathrm{x}, \tau/c)$ as $c \to \infty$, and then as $\tau \to \infty$, we obtain that the vertical velocity $w(\mathrm{x}, \tau/c)$ tends to $\tilde{w}(\mathrm{x}) = w^0(\mathrm{x}) + \hat{w}(\mathrm{x}) \Big(\hat{w}(\mathrm{x}) = (2\pi)^{-3} \int_{R^3} \exp\{-i(\mathbf{k}, \mathrm{x})\} \times (i\gamma - \varkappa/2)(k^2 + \varkappa^2/4)^{-1} Q^0(\mathbf{k})\, d\mathbf{k}\Big)$, i.e., is a solution of the equation

$$(\Delta - \varkappa^2/4)\hat{w} = -(\partial/\partial z - \varkappa/2)Q^0. \tag{3.7}$$

It is easy to confirm that (3.7) for $-\hat{w}$ agrees with (3.2) after going over to the original variable, so that $w_{\mathrm{p}}(\mathrm{x}) = -\hat{w}(\mathrm{x})$ and $\tilde{w}(\mathrm{x}) = w_{\mathrm{v}}(\mathrm{x})$.

The assertions of Theorem B' referring to the limit behavior of the density and pressure are proved analogously. The limit value of $\tilde{p}(\mathrm{x})$ is found in terms of the initial density $\rho^0(\mathrm{x})$ as the solution of the Poisson equation $(\Delta - \varkappa^2/4)\, w = (\partial/\partial z - \varkappa/2)\, Q^0$.

To prove Theorem A' we go over from (3.1) to the system (3.4) with constant coefficients. Using symbolic writing, we pass to the limit as $c \to \infty$. We then obtain

$$\begin{gathered}
Q \equiv 0,\ w \sim (\det B)^{-1}(-\omega(\gamma - \varkappa/2)Q^0 + \omega(k^2 - \varkappa^2/4)w^0 - \\
-g(k^2 - \gamma^2)\rho^0) = (\det B)^{-1}(\omega(k^2 - \varkappa^2/4)\tilde{w} - g(k^2 - \gamma^2)\rho^0), \\
\rho \sim (\det B)^{-1}(-\varkappa(\gamma - \varkappa/2)Q^0 + \varkappa(k^2 - \varkappa^2/4)w^0 + \omega(k^2 - \varkappa^2/4)\rho^0) = (\det B)^{-1}(k^2 - \varkappa^2/4)(\varkappa\tilde{w} + \omega\rho^0), \\
p \sim (\det B)^{-1}((\varkappa g + \omega^2)Q^0 - \varkappa g(\gamma + \varkappa/2)w^0 - \omega g(\gamma + \varkappa/2)\rho^0) = \\
= (\det B)^{-1}(\omega(k^2 - \varkappa^2/4)\tilde{p} - \varkappa g(\gamma + \varkappa/2)\tilde{w}) + Q^0/(k^2 - \varkappa^2/4) = \\
= -g(\gamma + \varkappa/2)(k^2 - \varkappa^2/4)^{-1}\rho,\ \det B = \omega^2 k^2 + N^2(k^2 - \gamma^2) - \omega^2\varkappa^2/4,\ N^2 = \varkappa g,
\end{gathered} \tag{3.8}$$

since $Q^0/(k^2 - \varkappa^2/4)$ yields no contribution for all $t > 0$. Let us note that (3.8) agrees with the formulas for the solution of the Cauchy problem in the initial data (w_{v}, ρ^0) found directly from (3.1').

4. In the problem of impact in an incompressible fluid ([3, Sec. 11] as well as [4]), fluid motion is considered under the action of impulsive mass forces $\mathbf{X}(\mathrm{x}, t)$ acting during a small time interval, where $\tilde{\mathbf{V}}(\mathrm{x}) = \int_0^\tau \mathbf{X}(\mathrm{x}, t)\, dt$, remains fixed as $\tau \to 0$. The Cauchy problem for (1.1) with zero initial perturbations $\tilde{\rho}(\mathrm{x})$, $\tilde{p}(\mathrm{x})$ can also be considered as the limit case of the impact problem since it is equivalent to the problem with zero initial data and δ-shaped mass forces $\mathbf{X}(\mathrm{x}, t)$ in the right sides of the equations for u_t, v_t, w_t. Let $\Phi(\tau, c, \mathrm{x}, t)$ denote the solution of the problem with zero initial data and mass forces acting for $0 \le t \le \tau$ (dependent on τ and c as on parameters). Let us define the double passage to the limit

$$\lim_{c\to\infty} \lim_{\tau\to 0} \Phi(\tau, c, \mathrm{x}, t), \tag{4.1}$$

where the inner limit is passage to the Cauchy problem with zero $\tilde{\rho}(\mathrm{x})$, $\tilde{p}(\mathrm{x})$ and nonzero $\tilde{u}(\mathrm{x})$, $\tilde{v}(\mathrm{x})$, $\tilde{w}(\mathrm{x})$, while the outer is passage to an incompressible fluid. The passage to the limit

$$\lim_{\tau\to 0} \lim_{c\to\infty} \Phi(\tau, c, \mathrm{x}, t) \tag{4.2}$$

is examined in [3, 4]. The inner limit is passage to an incompressible fluid while the outer describes the action of the mass forces.

Let us show that these limits agree. The fluid velocity after the action of impulsive forces

$$\mathbf{V}(\mathbf{x}) = \int_0^\tau \mathbf{X}(\mathbf{x}, t)\,dt - \rho_0^{-1}\int_0^\tau \operatorname{grad} p(\mathbf{x}, t)\,dt = \tilde{\mathbf{V}}(\mathbf{x}) - \rho_0^{-1}\operatorname{grad}\tilde{\omega}(\mathbf{x}), \tag{4.3}$$

is found in [3, Sec. 11, formula (1)], where $p(\mathbf{x}, t)$ is the pressure that occurs because of the action of the impulsive forces, while the function $\tilde{\omega}(\mathbf{x}) = \int_0^\tau p(\mathbf{x}, t)\,dt$ is called the impulsive pressure in [3]. The problem of determining the velocity $\mathbf{V}(\mathbf{x})$ reduces to evaluating $\tilde{\omega}(\mathbf{x})$ for which we use the fact that the divergence of the left side of (4.3) should be zero. Decomposing $\tilde{\mathbf{V}}(\mathbf{x})$ into potential and vortical parts according to (2.1a), we obtain $\mathbf{V}(\mathbf{x}) = \mathbf{v}_v(\mathbf{x})$ and $\tilde{\omega}(\mathbf{x}) = \varphi^0(\mathbf{x})$. Now agreement between the limits (4.1) and (4.2) results directly from Theorem A.

Let us note, however, that the problem considered above is more general than the impact problem, since although the initial velocity perturbation can be interpreted as the result of the action of the impulsive mass forces, the initial density and pressure perturbations are not successfully interpreted thus.

APPENDIX

The system (1.1) can be reduced to one equation for the vertical velocity w, say, which has the form (see (3.6)) $\partial^2/\partial t^2(\Delta w - \varkappa\partial w/\partial z) + N^2\Delta_h w - c^{-2}\partial^4 w/\partial t^4 = 0$ in the original variables. Utilizing the traditional Boussinesq approximation for simplicity, we discard the term $\varkappa\partial^3 w/\partial^2 t\partial z$ and write down the fundamental solution of the equation obtained:

$$L_c\mathscr{E}_c = (\partial^2/\partial t^2\Delta + N^2\Delta_h - c^{-2}\partial^4/\partial t^4)\mathscr{E}_c = \delta(\mathbf{x})\delta(t).$$

It is easy to show (compare with [5], formula (7.13)) that

$$\mathscr{E}_c = \frac{1}{8\pi^2 r}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \exp\left\{-i\omega t + \frac{i\omega r}{c}\sqrt{\frac{\omega^2 - N^2\cos^2\varphi}{\omega^2 - N^2}}\right\} \times \frac{d\omega}{\sqrt{\omega^2 - N^2}\sqrt{\omega^2 - N^2\cos^2\varphi}} \tag{A.1}$$

$$(\cos\varphi = z/r,\ r = (x^2 + y^2 + z^2)^{1/2}).$$

The solution of the Cauchy problem $L_c w = 0$ with the initial data $w|_{t=0} = w^0$, $\partial w/\partial t|_{t=0} = w^1$, $\partial^2 w/\partial t^2|_{t=0} = w^2$, $\partial^3 w/\partial t^3|_{t=0} = w^3$ is written

$$w = \partial/\partial t\,\square\,\mathscr{E}_c * w^0 + \square\,\mathscr{E}_c * w^1 - c^{-2}\partial\mathscr{E}_c/\partial t * w^2 - c^{-2}\mathscr{E}_c * w^3 \tag{A.2}$$

($\square = \Delta - c^{-2}\partial^2/\partial t^2$ is the D'Alembert operator. As $c \to \infty$, (A.2) becomes the known formula [2] for the solution of the Cauchy problem for the internal wave equation $w = \partial\mathscr{E}/\partial t * \Delta w^0 + \mathscr{E} * \Delta w^1$ ($\mathscr{E}$ is the fundamental solution of the internal wave equation). Returning to the "physical" initial data (w^0, ρ^0), we find $w = \partial\mathscr{E}/\partial t * \Delta w^0 - g\mathscr{E} * \Delta_h\rho^0$.

Setting $t = \tau/c$ in (A.1) and passing to the limit as $c \to \infty$, we obtain $\mathscr{E}_c(r, \varphi, \tau/c) \to \mathscr{E}_v(r, \tau) = (4\pi r)^{-1}(\tau - r)\,\theta(\tau)\theta(\tau - r)$, the fundamental solution of the ordinary wave equation integrated twice with respect to τ (θ is the Heaviside function).

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